Solution: Drawing Bound and Scattering State Wave Function
Warm-up:

1.

(a) No. The energy E is greater than the potential energy in region (i).
(b) No. The energy E is greater than the potential energy in region (ii).
(c) No. The energy E is greater than the potential energy in region (iii).
(d) No. The energy E is greater than the potential energy in regions (i) and (iii).
(e) *Correct. In all three regions, the total energy is greater than the potential energy.*

2.

(a) No. Region (i) is classically allowed.
(b) No. Region (ii) is classically allowed.
(c) No. Region (iii) is classically allowed.
(d) No. Regions (i) and (iii) are classically allowed.
(e) *Correct. All regions are classically allowed.*

3.

(a) No. It will not be rapidly decaying in region (ii) because the energy of the particle is greater than the potential energy.
(b) No. It will not be rapidly decaying in regions (i) and (iii) because the energy of the particle is greater than the potential energy.
(c) *Correct. It is classically allowed, so the wave function will be oscillatory in all regions.*
(d) The exact energies need not be known to determine the qualitative shape of the wave functions.
(e) No. There is a correct answer among the choices above.

4.

(I) Correct. The total energy is equal to the sum of potential and kinetic energies.
(II) Correct. If the potential energy is zero, the total energy is equal to the kinetic energy,
(III) Correct. $K=E+V_0$ in region (ii).

(a) No. There are other statements that are correct.
(b) No. There are other statements that are correct.
(c) No. There are other statements that are correct.
(d) No. There are other statements that are correct.
(e) *Correct. All three statements are correct statements.*

5.

- **(a)** No. Kinetic energy and momentum are directly related. As kinetic energy increases, momentum increases, and as kinetic energy decreases, momentum decreases.
- **(b)** No. Kinetic energy and momentum are directly related. As kinetic energy increases, momentum increases, and as kinetic energy decreases, momentum decreases.
- **(c)** *Correct.* $K_1=K_3<K_2$ *therefore* $p_1=p_3<p_2$.
- **(d)** No. Kinetic energy and momentum are directly related. As kinetic energy increases, momentum increases, and as kinetic energy decreases, momentum decreases.
- **(e)** No. Kinetic energy and momentum are directly related. As kinetic energy increases, momentum increases, and as kinetic energy decreases, momentum decreases.

6.

- **(a)** *Correct. This is the de Broglie relation. It is also dimensionally correct..*
- **(b)** No. This is dimensionally incorrect.
- **(c)** No. This is dimensionally incorrect.
- **(d)** No. This is dimensionally incorrect
- **(e)** No. This statement does not relate the wavelength to momentum.

7.

- **(a)** No. Momentum and wavelength are inversely related. The correct ordering can be determined from the order of momenta.
- **(b)** No. Momentum and wavelength are inversely related. The correct ordering can be determined from the order of momenta.
- **(c)** *Correct.* $p_1=p_3<p_2$ *therefore,* $\lambda_2 < \lambda_1 = \lambda_3$.
- **(d)** No. Momentum and wavelength are inversely related. The correct ordering can be determined from the order of momenta.
- **(e)** No. Momentum and wavelength are inversely related. The correct ordering can be determined from the order of momenta.

8. *Peter's argument is correct.* In the semi-classical approximation, one can use the momentum of the particle to determine its amplitude $|\Psi|$ by thinking of the electron as a moving particle. Then the probability density $|\Psi|^2$ can be determined by the relative time spent by the electron in a narrow range between x and $x+dx$.

9.

- **(a)** No. The amplitude and wavelength must change as the kinetic energy of the particle changes.
- **(b)** *Correct. The amplitude and wavelength in regions (i) and (iii) are the same. The amplitude and the wave length in region (ii) are both smaller than in regions (i) and (iii), which should be the case since the kinetic energy is higher.*
- **(c)** No. The wave function is discontinuous.
- **(d)** No. The middle part of the wave function does not cross the x-axis at all, whereas it should, like the wave functions in regions (i) and (iii).
- **(e)** No. The wave function is discontinuous and it is decaying in the classically allowed region.

10. *Peter's reasoning is correct.* Wave functions need to be continuous at all points. The points where the wave function go to zero need to be consistent in different regions.

11.

(I) $E<V(x)$
(II) $E>V(x)$
(III) $E<V(x)$

- **(a)** No. In region (i) the total energy is less than the potential energy but there is another region in which the total energy is less than the potential energy.
- **(b)** No. In region (ii) the total energy is greater than the potential energy.
- **(c)** No. In region (iii) the total energy is less than the potential energy but there is another region in which the total energy is less than the potential energy.
- **(d)** *Correct. In regions (i) and (iii) the total energy is less than the potential energy.*
- **(e)** No. There is a correct answer choice among the choices above.

12. Regions in which the total energy is less than the potential energy are classically forbidden regions. In light of the answer to question 11,

- **(a)** No. Region (i) is classically forbidden, but there is another region that is classically forbidden.
- **(b)** No. Region (ii) is classically allowed.
- **(c)** No. Region (iii) is classically forbidden but there is another region that is classically forbidden.
- **(d)** *Correct. Regions (i) and (iii) are classically forbidden.*
- **(e)** No. There is a correct answer among the choices above.

13. The wave function will be rapidly decaying in classically forbidden regions, and oscillatory in classically allowed regions.

(I) Region (i) is forbidden, the wave function will be rapidly decaying.
(II) Region (ii) is allowed, the wave function will be oscillatory.
(III) Region (iii) is forbidden, the wave function will be decaying exponentially.

- **(a)** No. This is a correct statement but there is another one that is correct.
- **(b)** No. This statement is correct but there is another one that is correct.
- **(c)** No. The wave function will not e oscillatory in region (iii).
- **(d)** *Correct. Statements (I) and (II) are correct statements.*
- **(e)** No. One of the statements (II) and (III) is incorrect.

14. When drawing the wave function, keep in mind that in classically forbidden regions the wave function will be rapidly decaying. In classically allowed regions the wave function will be oscillatory. Also remember that the wave function must be continuous at all points. The first derivative must be continuous at all points where the potential is finite.

Pretest:

1. Based on the semi-classical argument in the warm up part, we can conclude that the region where $x<a$ will be oscillatory. The region where $a<x<b$ is also oscillatory, but the wavelength is shorter and the amplitude smaller. The region where $x<b$ is the classically forbidden region, so the wave function will decay rapidly in that region.

2. All regions are classically allowed, so the wave function will be oscillatory everywhere. It will have relatively smaller amplitude in the region where the potential energy is higher, so the difference between the potential energy and the total energy is smaller.

Tutorial:

1.

(I) Correct. The potential energy is less than the total energy.

(II) Correct. (i) and (iii) are classically allowed so the wave function will be oscillatory.

(III) (ii) is classically forbidden, the wave function will be rapidly decaying.

- **(a)** No. (I) is a correct statement but there are other ones that are correct.
- **(b)** No. (II) is a correct statement but there are other ones that are correct.
- **(c)** No. (III) is incorrect, the wave function will not be oscillatory in region (ii).
- **(d)** *Correct. (I) and (II) are correct statements.*
- **(e)** No. One of the statements (I) and (III) is incorrect.

2.

(I) Correct. The total energy and the potenetial energy are the same in region s(i) and (iii), so the kinetic energy is also the same.

(II) Correct. Since the kinetic energies are the same, the momenta are also the same.

(III) Correct. Since the momenta are the same, by the de Broglie relation, the wave lengths are the same as well.

- **(a)** No. Statement (I) is correct but there are other ones that are correct.
- **(b)** No. Statement (II) is correct but there are other ones that are correct.
- **(c)** No. Statements (I) and (II) are correct but there are other ones that are correct.
- **(d)** No. Statements (II) and (III) are correct but there are other ones that are correct.
- **(e)** *Correct. All of the above statements are correct statements.*

3.

- **(a)** *Correct. The amplitudes are equal because the momenta are equal.*
- **(b)** No. The amplitudes are inversely related with momentum. If the momenta of the particle in two regions are the same, so should be the amplitudes of the wave function.
- **(c)** No. The amplitudes are inversely related with momentum. If the momenta of the particle in two regions are the same, so should be the amplitudes of the wave function.
- **(d)** No. The amplitudes are inversely related with momentum. If the momenta of the particle in two regions are the same, so should be the amplitudes of the wave function.
- **(e)** No. The amplitudes are inversely related with momentum. If the momenta of the particle in two regions are the same, so should be the amplitudes of the wave function.

4.

- **(a)** The wave function needs to be decaying exponentially in region (ii).
- **(b)** *Correct. The wave function is decaying exponentially in region (ii). The amplitudes and wavelengths in regions (i) and (iii) are the same. The wave function is continuous, and so is its first derivative.*
- **(c)** The wave function is discontinuous. It is decaying in allowed regions and is oscillatory in the forbidden region.
- **(d)** The first derivative of the wave function is discontinuous at $x=a$.
- **(e)** The wave function and its first derivative are discontinuous at $x=0$.

5. *Jim is right*. There is no reason why the wave function should prefer decaying in one direction over the other when no direction is preferred.

6. *Alex is wrong.* If it were a wave packet moving toward the barrier from the right he would be correct, but this is an eigenstate for an electron with fixed energy. The wave function merely represents the probability amplitude of finding the electron at a position. The beam moving toward the barrier from the left is a very common example in quantum mechanics classes, that is why Alex remembers that wave function.

7.

(I) Correct. The regions (i) and (iii) are classically forbidden because the potential energy exceeds the total energy.

(II) Correct. The electron cannot be found in regions (i) and (iii) because the potential is infinite in those regions.

(III) No. Since the potential is infinite, the wave function will be zero abruptly, and not be decaying.

- **(a)** No. Statement (I) is correct but there are other ones that are correct.
- **(b)** No. Statement (II) is correct, but there are other ones that are correct.
- **(c)** No. The electron cannot be found in the region where the potential energy is infinite.
- **(d)** *Correct. Statements (I) and (II) are correct, and (III) is not.*
- **(e)** No. One of the statements (I) and (III) is incorrect.

8. The wave function is zero in regions where the potential energy is infinite. The wave function is oscillatory in regions where the total energy is greater than the total energy. Remember that the wave function must be continuous at all points. The first derivative (slope) of the wave function must be continuous at all points where the potential energy is finite. In this case, the first derivative will not be continuous at $x=0$ and $x=a$.

9. The derivative of the wave function is not continuous at $x=0$ and $x=a$ because the potential goes to infinity at those points abruptly. The wave function goes to zero abruptly, so the derivative of the wave function is not continuous at those points.

10. *Debbie's explanation is correct.* In quantum mechanics, particles can have non zero probability of being found in the classically forbidden regions. Their wave functions must be rapidly decaying in those regions. An exception is when the potential energy is not only greater than the total energy but also infinite. In such cases (which are artificial cases used to approximate very very large potentials), the wave function does become zero instantly, and the slope is discontinuous at the boundaries.

11. $(-\infty, a)$ and (b, ∞) are classically forbidden.

12. The electron cannot be found in region (i) because the potential is infinite.

13. The electron can be found in region (ii), the total energy is greater than the potential energy.

14. The electron can be found in region (iii) although it is classically forbidden because the potential energy is finite.

15. The kinetic energy is positive in the region (ii) only

16. The wave function is always zero in region (i), because the potential is infinite.

17. The wave function will be rapidly decaying in region (iii) because the potential energy is greater than the total energy, but it is finite.

18. The wave function will be oscillatory in region (ii) because the potential energy is less than the total energy so the particle is classically allowed to be in region (ii).

19. The kinetic energy in region (ii), where the wave function is oscillatory is decreasing with increasing *x*.

20. Since the kinetic energy is increasing, the momentum is also decreasing.

21. The de Broglie relation is, $p = \frac{h}{\lambda}$, where h is the Planck constant and λ is the wavelength of the particle.

22. Since the momentum is decreasing and the wavelength is inversely proportional to momentum, the wavelength will be increasing with increasing *x*.

23. Similarly, since the smaller the momentum the larger the amplitude, the amplitude will also be increasing with increasing *x*.

24. The wave function must be zero at positions where the potential energy is infinite. The wave function is zero at all points $x<a$. In region (ii) the wave function will be oscillatory, but the wave length and amplitude will not be constant. As the kinetic

energy decreases, the wave length and amplitude will increase. Beyond $x=b$, the wave function will decay exponentially. The particle can be found at $x>b$ but not at $x = \infty$.

25. The wave function must be continuous everywhere.

26. The first derivative (slope) of the wave function must be continuous everywhere except $x=a$. At $x=a$ the potential energy becomes infinite instantly, so the wave function must go to zero instantly. The first derivative can be discontinuous at that point.

27. The wave function needs to be rapidly decaying in regions where the potential energy is higher than the total energy of the particle. In regions where the total energy is greater than the potential energy, the wave function will be oscillatory. There are two such regions, with different potential energies. The amplitude and wavelength will be smaller where the kinetic energy is larger.

28. The wave function will be oscillatory in the region where the total energy is greater than the potential energy, and rapidly decaying in the region where the potential energy is greater than the total energy.

29. The wave functions in the previous two questions must be continuous at all points. The first derivative of the wave functions must also be continuous at all points, since there is no point where the potential goes to infinity abruptly.

Posttest:

1. This is very similar to the V shaped potential energy diagram for which you drew the wave function. The wave function will decay rapidly at $x<a$. Then it will be oscillatory in the region between $x=a$ and $x=b$. The wave length and the amplitude will be decreasing until $x=b$. The wave function will abruptly go to zero at $x=b$ and the first derivative will be discontinuous at that point. The potential energy goes to infinity abruptly.
2. For the first figure, the regions $x<0$ and $x>b$ are regions in which the wave function is decaying. In the middle two regions the wave function will be oscillatory. The wave length and amplitude will be greater between $x=0$ and $x=a$ because the kinetic energy is smaller there.

   For the second figure, the wave function will be oscillatory everywhere. The relative amplitudes and wave lengths can be determined by the semi-classical approximation. The wave length and amplitude will be greatest at $x<0$ and $x>b$. The largest wavelength and amplitude will occur at $0<x<a$. The smallest will be $a<x<b$.